$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$ $E_b = \sigma T^4$

ATKINSON SCIENCE

$e^{i\pi} = -1$ $\frac{u}{u_\tau} = \frac{1}{\kappa}\ln\frac{y u_\tau}{\nu} + C$

THEORY GUIDE

Equations of Fluid Flow

Momentum Equations in Cylindrical Coordinates

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Atkinson Science welcomes your comments on this Theory Guide. Please send an email to keith.atkinson@atkinsonscience.co.uk.

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1 r component momentum equation

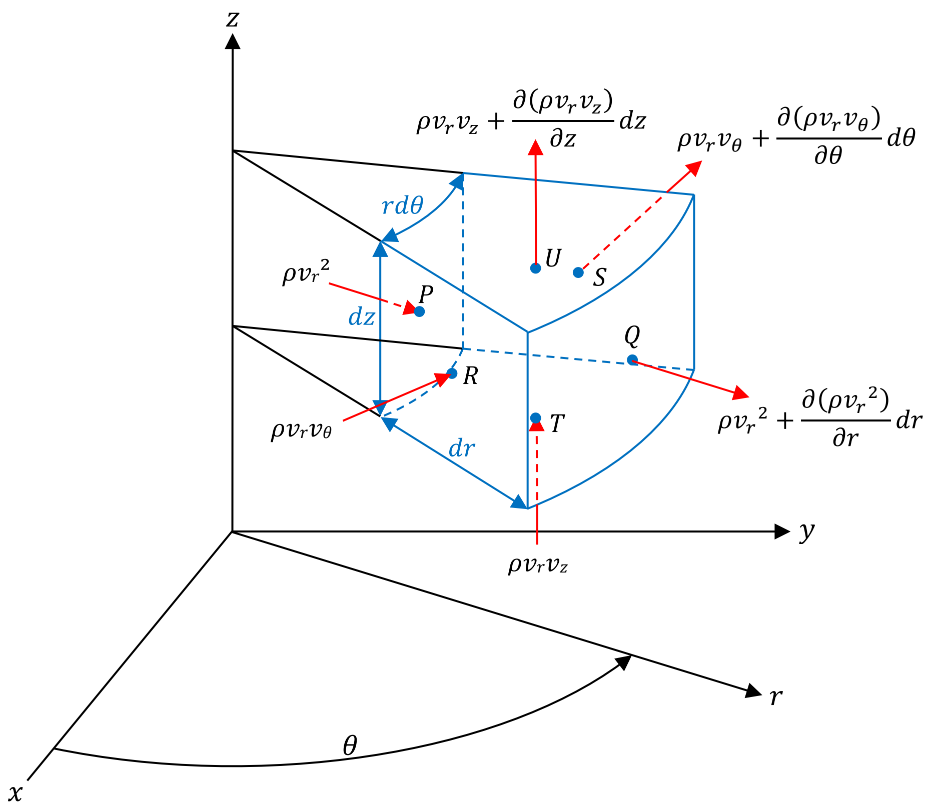
1.1 Control volume analysis

To derive the equation for the r component of momentum in cylindrical coordinates we consider the properties of the flow into and out of the infinitesimal control volume (CV) shown in Figure 1. Applying Newton's second law of motion to the CV:

Rate of increase of r component momentum of fluid in CV	=	Rate of flow of r component momentum into CV	−	Rate of flow of r component momentum out of CV	+	Sum of r components of forces applied to fluid in CV		(1.1)

To derive the equations for the θ and z components of momentum, we replace r in (1.1) with θ and z, respectively.

Figure 1 r component momentum fluxes at the surfaces of a control volume



1.2 Transient and convection terms

The r component of momentum in the CV is equal to the r component of velocity, v_r, times the mass of fluid in the CV, $\rho\, dr\, rd\theta\, dz$; that is, $\rho\, v_r\, dr\, rd\theta\, dz$. The rate of increase of r component momentum with time (the first term in Eq. (1.1)), is therefore

$$\frac{\partial(\rho v_r)}{\partial t} dr\, rd\theta\, dz \qquad (1.2)$$

r component momentum may enter or leave through any of the faces P to U in Figure 1, transported by the mass flow through the faces.

The rate of flow of r component momentum through the face perpendicular to the r direction whose centre is P is v_r times the mass flow through the face, $\rho\, v_r\, rd\theta\, dz$; that is,

$$\rho {v_r}^2\, rd\theta\, dz$$

The rate of flow of r component momentum through the opposite face whose centre is Q is

$$\left(\rho {v_r}^2 + \frac{\partial(\rho {v_r}^2)}{\partial r} dr\right)(r + dr) d\theta\, dz$$

and so the net rate of flow of r component momentum out of the CV through the faces with centres P and Q is

$$\left(\rho {v_r}^2 + \frac{\partial(\rho {v_r}^2)}{\partial r} dr\right)(r + dr) d\theta\, dz - (\rho {v_r}^2)\, rd\theta\, dz =$$

$$\frac{\partial(\rho {v_r}^2)}{\partial r} dr\, rd\theta\, dz + \rho {v_r}^2\, dr\, d\theta\, dz + \frac{\partial(\rho {v_r}^2)}{\partial r} dr^2\, d\theta\, dz$$

We can neglect the term in dr^2, so the net rate of flow of r component momentum out of the CV through the faces with centres P and Q is

$$\frac{\partial(\rho {v_r}^2)}{\partial r} dr\, rd\theta\, dz + \rho {v_r}^2\, dr\, d\theta\, dz \qquad (1.3)$$

The rate of flow of r component momentum through the face perpendicular to the θ direction whose centre is R is v_r times the mass flow through the face, $\rho\, v_\theta\, dr\, dz$; that is,

$$(\rho v_r v_\theta)\; dr\; dz$$

The corresponding rate of flow of r component momentum out of the face with centre S is

$$\left(\rho v_r v_\theta + \frac{\partial(\rho v_r v_\theta)}{\partial \theta} d\theta\right) dr\; dz$$

so the net rate of flow of r component momentum out of the CV through the faces with centres R and S is

$$\frac{\partial(\rho v_r v_\theta)}{\partial \theta} dr\; d\theta\; dz \qquad (1.4)$$

The rate of flow of r component momentum through the face perpendicular to the z direction with centre T is v_r times the mass flow through the face. The area of the face is

$$dr\;(r + ½dr)d\theta$$

so the rate of flow of mass through the face is

$$(\rho v_z)\; dr\;(r + ½dr)d\theta$$

and the rate of flow of r component momentum through the face is

$$(\rho v_r v_z)\; dr\;(r + ½dr)d\theta$$

The corresponding rate of flow of r component momentum out of the face with centre U is

$$\left(\rho v_r v_z + \frac{\partial(\rho v_r v_z)}{\partial z} dz\right) dr\;(r + ½dr)d\theta$$

so the net rate of flow of r component momentum out of the CV through the faces with centres T and U is

$$\frac{\partial(\rho v_r v_z)}{\partial z} dr\;(r + ½dr)d\theta\; dz = \frac{\partial(\rho v_r v_z)}{\partial z} dr\; rd\theta\; dz + \frac{\partial(\rho v_r v_z)}{\partial z} ½dr^2\; d\theta\; dz$$

We can neglect the term in dr^2, so the net rate of flow of r component momentum out of the CV is

$$\frac{\partial(\rho v_r v_z)}{\partial z} dr\; rd\theta\; dz \qquad (1.5)$$

Adding together (1.3), (1.4) and (1.5), the sum of the net rates of outflow of r component momentum is

$$\left[\frac{\partial(\rho {v_r}^2)}{\partial r} + \frac{\rho {v_r}^2}{r} + \frac{1}{r}\frac{\partial(\rho v_r v_\theta)}{\partial \theta} + \frac{\partial(\rho v_r v_z)}{\partial z}\right] dr\; rd\theta\; dz$$

Finally, we can combine the first and second terms in the brackets into one:

$$\left[\frac{1}{r}\frac{\partial(r\rho {v_r}^2)}{\partial r} + \frac{1}{r}\frac{\partial(\rho v_r v_\theta)}{\partial \theta} + \frac{\partial(\rho v_r v_z)}{\partial z}\right] dr\; rd\theta\; dz \qquad (1.6)$$

1.3 Body force terms

There are two types of forces acting on the fluid in the CV: body forces and surface forces. The simplest example of a body force is the gravitational force. The fluid in the CV is subject to a gravitational force equal to g, the acceleration due to gravity, times the mass of the fluid, $\rho\; dr\; rd\theta\; dz$; that is, $\rho g\; dr\; rd\theta\; dz$. A body force is a vector, so in general it has three components f_r, f_θ, f_z per unit mass. The body force acting in the r coordinate direction is

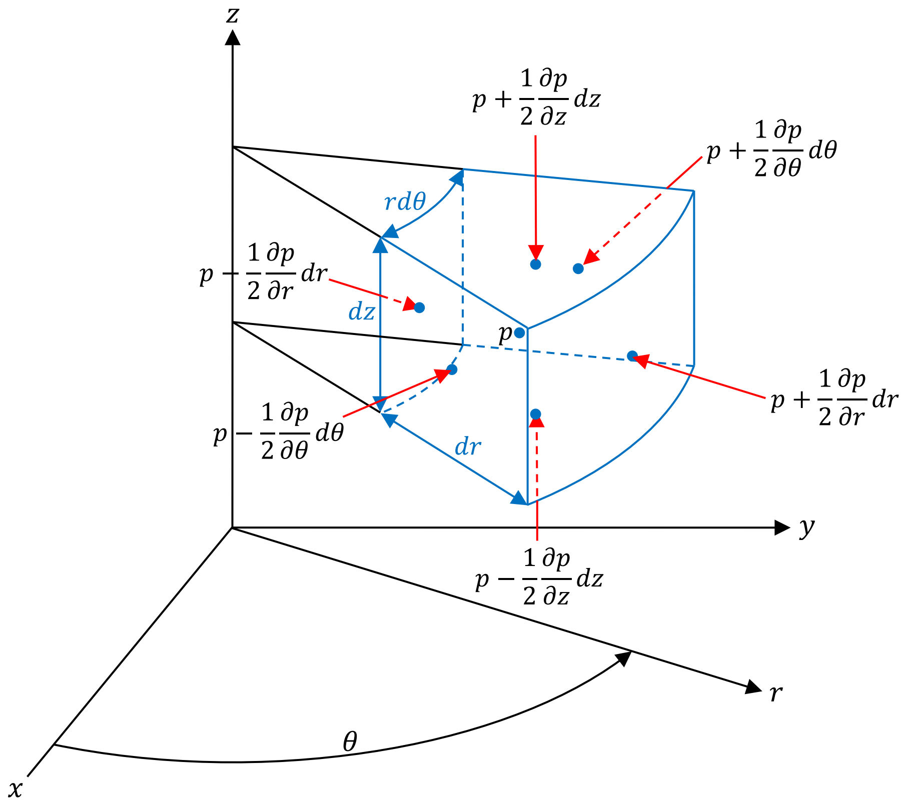
$$\rho f_r\; dr\; rd\theta\; dz \qquad (1.7)$$

1.4 Pressure term

1.4.1 Method 1

Figure 2 shows the pressure acting on the control volume. By definition, a positive pressure acts inwards.

Figure 2 Pressure acting on the control volume



The net force in the r direction due to the pressure is

$$\left(p - \frac{1}{2}\frac{\partial p}{\partial r} dr\right) r d\theta\, dz - \left(p + \frac{1}{2}\frac{\partial p}{\partial r} dr\right)(r + dr) d\theta\, dz$$

$$+\left(p - \frac{1}{2}\frac{\partial p}{\partial \theta} d\theta\right) dr\, dz\, \sin\frac{d\theta}{2} + \left(p + \frac{1}{2}\frac{\partial p}{\partial \theta} d\theta\right) dr\, dz\, \sin\frac{d\theta}{2}$$

$$= -\frac{1}{2}\frac{\partial p}{\partial r} dr\, r d\theta\, dz\; - p\, dr\, d\theta\, dz - \frac{1}{2}\frac{\partial p}{\partial r} dr\, r d\theta\, dz - \frac{1}{2}\frac{\partial p}{\partial r} dr^2\, d\theta\, dz$$

$$+p\, dr\, dz\, \sin\frac{d\theta}{2} - \frac{1}{2}\frac{\partial p}{\partial \theta} dr\, d\theta\, dz\, \sin\frac{d\theta}{2} + p\, dr\, dz\, \sin\frac{d\theta}{2} + \frac{1}{2}\frac{\partial p}{\partial \theta} dr\, d\theta\, dz\, \sin\frac{d\theta}{2}$$

For small θ we can approximate $\sin\theta$ as θ in radians. With this approximation and after neglecting the dr^2 term, the net force in the r direction due to the pressure is

$$-\frac{1}{2}\frac{\partial p}{\partial r}dr\, rd\theta\, dz\; - p\, dr\, d\theta\, dz - \frac{1}{2}\frac{\partial p}{\partial r}dr\, rd\theta\, dz$$

$$+p\, dr\, dz\,\frac{d\theta}{2} - \frac{1}{2}\frac{\partial p}{\partial \theta}dr\, d\theta\, dz\,\frac{d\theta}{2} + p\, dr\, dz\frac{d\theta}{2} + \frac{1}{2}\frac{\partial p}{\partial \theta}dr\, d\theta\, dz\,\frac{d\theta}{2}$$

After neglecting the $d\theta^2$ terms and summing the remaining terms, the net force is

$$-\frac{\partial p}{\partial r}dr\, rd\theta\, dz \qquad (1.8)$$

1.4.2 Method 2

The pressure p is a scalar variable. When the x momentum equation is expressed in Cartesian coordinates, the pressure term is equal to minus the gradient of the pressure in the x coordinate direction, or minus the $\mathbf{i}$ component of the gradient $\boldsymbol{\nabla}p$,

$$-\boldsymbol{\nabla}p = -\left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)p$$

$$= -\frac{\partial p}{\partial x}\mathbf{i} - \frac{\partial p}{\partial y}\mathbf{j} - \frac{\partial p}{\partial z}\mathbf{k}$$

In orthogonal curvilinear coordinates the gradient operator $\boldsymbol{\nabla}$ is

$$\boldsymbol{\nabla} = \frac{\mathbf{e}_1}{h_1}\frac{\partial}{\partial u_1} + \frac{\mathbf{e}_2}{h_2}\frac{\partial}{\partial u_2} + \frac{\mathbf{e}_3}{h_3}\frac{\partial}{\partial u_3}$$

where $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are unit vectors tangent to the coordinate lines u_1, u_2, u_3, respectively, and h_1, h_2, h_3 are scale factors (see Chapter 7, Problem 18 of Ref. [1]). The cylindrical coordinate system is an orthogonal system, with

$$u_1 = r, \quad u_2 = \theta, \quad u_3 = z$$

$$\mathbf{e}_1 = \mathbf{e}_r, \quad \mathbf{e}_2 = \mathbf{e}_\theta, \quad \mathbf{e}_3 = \mathbf{e}_z$$

$$h_1 = h_r = 1, \quad h_2 = h_\theta = r, \quad h_3 = h_z = 1$$

(see Chapter 7, Problem 7 of Ref. [1]) and so in cylindrical coordinates the gradient operator is

$$\boldsymbol{\nabla} = \frac{\partial}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\mathbf{e}_\theta + \frac{\partial}{\partial z}\mathbf{e}_z$$

and the gradient of the pressure is

$$-\boldsymbol{\nabla}p = -\left(\frac{\partial p}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial p}{\partial \theta}\mathbf{e}_\theta + \frac{\partial p}{\partial z}\mathbf{e}_z\right)$$

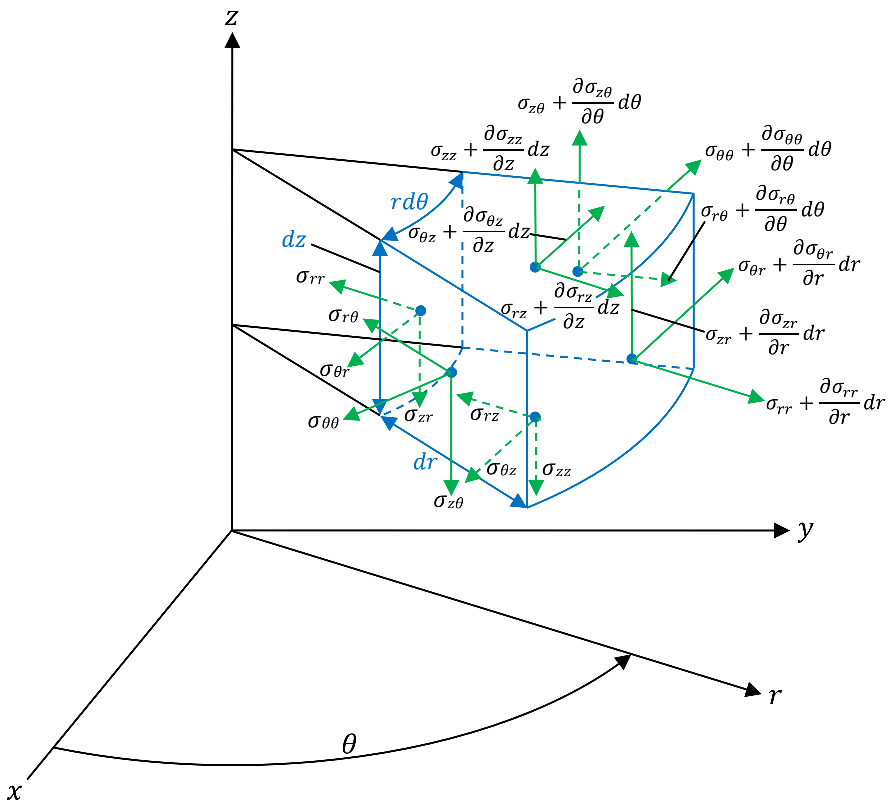
The pressure term in the r component momentum equation is then

$$-\frac{\partial p}{\partial r} \qquad (1.8)$$

1.5 Viscous stress terms

The viscous normal stresses and shear stresses acting on the control volume are shown in Figure 3. We define the different components of viscous normal stress and viscous shear stress as shown in Figure 3. The first subscript of the symbol σ represents the direction of the stress and the second subscript represents the direction of the surface normal.

Figure 3 Viscous stresses on the control volume



Normal stresses

By convention, an *outward* normal stress acting on the CV is positive.

Shear stresses

By convention, the shear stresses are taken as positive on the faces farthest from the origin and negative on the faces nearest to the origin. Thus a shear stress $\sigma_{r\theta}$ acts in the positive r direction on the invisible (upper) face perpendicular to the θ axis and a corresponding shear stress acts in the negative r direction on the visible (lower) face perpendicular to the θ axis.

1.5.1 σ_{rr} stress

Referring to Figure 3, the net force in the r direction due to the stress σ_{rr} is

$$\left(\sigma_{rr} + \frac{\partial \sigma_{rr}}{\partial r} dr\right)(r + dr) d\theta\, dz - \sigma_{rr} r d\theta\, dz$$

$$= \sigma_{rr}\, dr\, d\theta\, dz + \frac{\partial \sigma_{rr}}{\partial r} dr\, r d\theta\, dz\; + \frac{\partial \sigma_{rr}}{\partial r} dr^2\, d\theta\, dz$$

We can neglect the term in dr^2, so the net force in the r direction due to the stress σ_{rr} is

$$\sigma_{rr}\, dr\, d\theta\, dz + \frac{\partial \sigma_{rr}}{\partial r} dr\, r d\theta\, dz \qquad (1.9)$$

1.5.2 $\sigma_{r\theta}$ stress

The net force in the r direction due to the stress $\sigma_{r\theta}$ is

$$\left(\sigma_{r\theta} + \frac{\partial \sigma_{r\theta}}{\partial \theta} d\theta\right) dr\, dz\; \cos\frac{d\theta}{2} - \sigma_{r\theta} dr\, dz\; \cos\frac{d\theta}{2}$$

For small θ we can approximate $\cos\theta$ as $1 - \theta$ in radians. The net force in the r direction is then

$$\sigma_{r\theta}\, dr\, dz - \sigma_{r\theta}\, dr\, dz \frac{d\theta}{2} + \frac{\partial \sigma_{r\theta}}{\partial \theta} d\theta\, dr\, dz - \frac{\partial \sigma_{r\theta}}{\partial \theta} d\theta\, dr\, dz \frac{d\theta}{2} - \sigma_{r\theta} dr\, dz + \sigma_{r\theta} dr\, dz \frac{d\theta}{2}$$

$$= \frac{\partial \sigma_{r\theta}}{\partial \theta} dr\, d\theta\, dz - \frac{\partial \sigma_{r\theta}}{\partial \theta} dr\, dz \frac{d\theta^2}{2}$$

We can neglect the term in $d\theta^2$, so the net force in the r direction due to the stress $\sigma_{r\theta}$ is

$$= \frac{\partial \sigma_{r\theta}}{\partial \theta} dr\, d\theta\, dz \qquad (1.10)$$

1.5.3 σ_{rz} stress

The net force in the r direction due to the stress σ_{rz} is

$$\left(\sigma_{rz} + \frac{\partial \sigma_{rz}}{\partial z} dz\right) dr\, (r + ½dr) d\theta - \sigma_{rz} dr\, (r + ½dr) d\theta$$

$$= \sigma_{rz}\, dr\, r d\theta + \sigma_{rz}\, ½dr^2\, d\theta + \frac{\partial \sigma_{rz}}{\partial z} dr\, r d\theta\, dz + \frac{\partial \sigma_{rz}}{\partial z} ½dr^2\, d\theta\, dz - \; \sigma_{rz}\, dr\, r d\theta - \sigma_{rz}\, ½dr^2\, d\theta$$

$$= \frac{\partial \sigma_{rz}}{\partial z}\; dr\, r d\theta\, dz + \frac{\partial \sigma_{rz}}{\partial z} ½dr^2\, d\theta\, dz$$

We can neglect the term in dr^2, so the net force in the r direction due to the stress σ_{rz} is

$$\frac{\partial \sigma_{rz}}{\partial z}\; dr\, r d\theta\, dz \qquad (1.11)$$

1.5.4 $\sigma_{\theta\theta}$ stress

The net force in the r direction due to the stress $\sigma_{\theta\theta}$ is

$$-\left(\sigma_{\theta\theta}+\frac{\partial\sigma_{\theta\theta}}{\partial\theta}d\theta\right)dr\;dz\;\sin\frac{d\theta}{2}-\sigma_{\theta\theta}\;dr\;dz\;\sin\frac{d\theta}{2}$$

For small θ we can approximate $\sin\theta$ as θ in radians. With this approximation the net force in the r direction due to the stress $\sigma_{\theta\theta}$ is

$$-\sigma_{\theta\theta}\;dr\;dz\;\frac{d\theta}{2}-\frac{\partial\sigma_{\theta\theta}}{\partial\theta}\;dr\;dz\;\frac{d\theta^2}{2}-\sigma_{\theta\theta}\;dr\;dz\;\frac{d\theta}{2}$$

After neglecting the term in $d\theta^2$, the net force in the r direction is

$$-\sigma_{\theta\theta}\;dr\;d\theta\;dz \qquad (1.12)$$

Adding together the expressions (1.9), (1.10), (1.11) and (1.12), the sum of the net forces in the r direction due to the viscous stresses is

$$\left[\frac{\sigma_{rr}}{r}+\frac{\partial\sigma_{rr}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta}+\frac{\partial\sigma_{rz}}{\partial z}-\frac{\sigma_{\theta\theta}}{r}\right]dr\;rd\theta\;dz \qquad (1.13)$$

1.6 r component momentum equation in terms of stress

By substituting expressions (1.2), (1.6), (1.7), (1.8) and (1.13) into Eq. (1.1) and dividing by the volume of the CV, $dr\ rd\theta\ dz$, we obtain the r component momentum equation in terms of stress:

$$\frac{\partial(\rho v_r)}{\partial t}+\frac{1}{r}\frac{\partial(r\rho {v_r}^2)}{\partial r}+\frac{1}{r}\frac{\partial(\rho v_r v_\theta)}{\partial \theta}+\frac{\partial(\rho v_r v_z)}{\partial z}$$
$$=-\frac{\partial p}{\partial r}+\frac{\partial \sigma_{rr}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta}+\frac{\partial \sigma_{rz}}{\partial z}+\frac{\sigma_{rr}}{r}-\frac{\sigma_{\theta\theta}}{r}+\rho f_r \qquad (1.14)$$

1.7 Constitutive equations

For a variable-density Newtonian viscous fluid, the viscous normal stresses σ_{rr}, $\sigma_{\theta\theta}$, σ_{zz} and the viscous shear stresses $\sigma_{r\theta}$, $\sigma_{\theta r}$, σ_{rz}, σ_{zr}, $\sigma_{\theta z}$, $\sigma_{z\theta}$ are given by

$$\sigma_{rr} = 2\mu\frac{\partial v_r}{\partial r} + \lambda\boldsymbol{\nabla}\cdot\mathbf{V}$$

$$\sigma_{\theta\theta} = 2\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) + \lambda\boldsymbol{\nabla}\cdot\mathbf{V}$$

$$\sigma_{zz} = 2\mu\frac{\partial v_z}{\partial z} + \lambda\boldsymbol{\nabla}\cdot\mathbf{V}$$

$$\sigma_{r\theta} = \sigma_{\theta r} = \mu\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right)$$

$$\sigma_{rz} = \sigma_{zr} = \mu\left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right)$$

$$\sigma_{\theta z} = \sigma_{z\theta} = \mu\left(\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta}\right)$$

In these equations μ is the *shear viscosity* or *first viscosity* and λ is the *volume viscosity* or *bulk viscosity*. The *second viscosity* ζ is defined by

$$\zeta = \lambda + \frac{2}{3}\mu$$

The second viscosity ζ is often assumed to be zero, making the volume viscosity λ equal to $-2\mu/3$.

We can express the divergence $\nabla \cdot \mathbf{V}$ in orthogonal curvilinear coordinates as follows

$$\nabla \cdot \mathbf{V} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(V_1 h_2 h_3) + \frac{\partial}{\partial u_2}(V_2 h_3 h_1) + \frac{\partial}{\partial u_3}(V_3 h_1 h_2)\right] \qquad (1.15)$$

where (u_1, u_2, u_3) is an orthogonal curvilinear coordinate system, $V_1 = V_1(u_1, u_2, u_3)$, $V_2 = V_2(u_1, u_2, u_3)$, $V_3 = V_3(u_1, u_2, u_3)$, and h_1, h_2, h_3 are *scale factors*.

We can show that the cylindrical coordinate system is orthogonal (see Chapter 7, Problem 3 of Ref. [1]). We have

$$u_1 = r,\ u_2 = \theta,\ u_3 = z$$

$$V_1 = v_r,\ V_2 = v_\theta,\ V_3 = v_z$$

$$h_1 = h_r = 1,\ h_2 = h_\theta = r,\ h_3 = h_z = 1$$

(see Chapter 7, Problem 7 of Ref. [1]). Substituting these expressions into (1.15) gives

$$\nabla \cdot \mathbf{V} = \frac{1}{r}\left[\frac{\partial (r v_r)}{\partial r} + \frac{\partial (v_\theta)}{\partial \theta} + \frac{\partial (r v_z)}{\partial z}\right]$$

$$= \frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} \qquad (1.16)$$

1.8 r component momentum equation in terms of velocity

Substituting the equations for σ_{rr}, $\sigma_{r\theta}$, σ_{rz}, $\sigma_{\theta\theta}$ into the Eq. (1.14) gives

$$\frac{\partial(\rho v_r)}{\partial t}+\frac{1}{r}\frac{\partial(r\rho {v_r}^2)}{\partial r}+\frac{1}{r}\frac{\partial(\rho v_r v_\theta)}{\partial \theta}+\frac{\partial(\rho v_r v_z)}{\partial z}=-\frac{\partial p}{\partial r}+\frac{\partial}{\partial r}\left[2\mu\frac{\partial v_r}{\partial r}+\lambda\nabla\cdot\mathbf{V}\right]$$

$$+\frac{1}{r}\frac{\partial}{\partial\theta}\left[\mu\left(\frac{1}{r}\frac{\partial v_r}{\partial\theta}+\frac{\partial v_\theta}{\partial r}-\frac{v_\theta}{r}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v_r}{\partial z}+\frac{\partial v_z}{\partial r}\right)\right]$$

$$+\frac{1}{r}\left[2\mu\frac{\partial v_r}{\partial r}+\lambda\nabla\cdot\mathbf{V}\right]-\frac{1}{r}\left[2\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_r}{r}\right)+\lambda\nabla\cdot\mathbf{V}\right]+\rho f_r$$

After simplifying, we obtain the r component momentum equation for three-dimensional unsteady compressible flow in cylindrical coordinates:

$$\frac{1}{r}\frac{\partial(r\rho {v_r}^2)}{\partial r}+\frac{1}{r}\frac{\partial(\rho v_r v_\theta)}{\partial \theta}+\frac{\partial(\rho v_r v_z)}{\partial z}=-\frac{\partial p}{\partial r}+\frac{\partial}{\partial r}\left[2\mu\frac{\partial v_r}{\partial r}+\lambda\nabla\cdot\mathbf{V}\right]$$

$$+\frac{1}{r}\frac{\partial}{\partial\theta}\left[\mu\left(\frac{1}{r}\frac{\partial v_r}{\partial\theta}+\frac{\partial v_\theta}{\partial r}-\frac{v_\theta}{r}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v_r}{\partial z}+\frac{\partial v_z}{\partial r}\right)\right]$$

$$+\frac{2\mu}{r}\left[\frac{\partial v_r}{\partial r}-\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}-\frac{v_r}{r}\right]+\rho f_r \qquad (1.17)$$

2 θ component momentum equation

2.1 Transient and convection terms

The θ component of momentum in the CV of Figure 1 is equal to the θ component of velocity, v_θ, times the mass of fluid in the CV, $\rho\ dr\ rd\theta\ dz$; that is, $\rho\ v_\theta\ dr\ rd\theta\ dz$. The rate of increase of θ component momentum with time (the first term in Eq. (1.1)), is therefore

$$\frac{\partial(\rho v_\theta)}{\partial t} dr\ rd\theta\ dz \qquad (2.1)$$

The rate of flow of θ component momentum through the face perpendicular to the r direction whose centre is P is v_θ times the mass flow through the face, $\rho\ v_r\ rd\theta\ dz$; that is,

$$\rho v_r v_\theta\ rd\theta\ dz$$

The rate of flow of θ component momentum through the opposite face whose centre is Q is

$$\left(\rho v_r v_\theta + \frac{\partial(\rho v_r v_\theta)}{\partial r} dr\right)(r + dr)d\theta\ dz$$

and so the net rate of flow of θ component momentum out of the CV through the faces with centres P and Q is

$$\left(\rho v_r v_\theta + \frac{\partial(\rho v_r v_\theta)}{\partial r} dr\right)(r + dr)d\theta\ dz - (\rho v_r v_\theta)\ rd\theta\ dz =$$

$$\frac{\partial(\rho v_r v_\theta)}{\partial r} dr\ rd\theta\ dz + \rho v_r v_\theta dr\ d\theta\ dz + \frac{\partial(\rho v_r v_\theta)}{\partial r} dr^2\ d\theta\ dz$$

We can neglect the term in dr^2, so the net rate of flow of θ component momentum out of the CV through the faces with centres P and Q is

$$\frac{\partial(\rho v_r v_\theta)}{\partial r} dr\ rd\theta\ dz + \rho v_r v_\theta\ dr\ d\theta\ dz \qquad (2.2)$$

The rate of flow of θ component momentum through the face perpendicular to the θ direction whose centre is R is v_θ times the mass flow through the face, $\rho\, v_\theta\, dr\, dz$; that is,

$$\rho v_\theta{}^2\, dr\, dz$$

The corresponding rate of flow of θ component momentum out of the face with centre S is

$$\left(\rho v_\theta{}^2 + \frac{\partial(\rho v_\theta{}^2)}{\partial\theta} d\theta\right) dr\, dz$$

so the net rate of flow of θ component momentum out of the CV through the faces with centres R and S is

$$\frac{\partial(\rho v_\theta{}^2)}{\partial\theta} dr\, d\theta\, dz \qquad (2.3)$$

The rate of flow of θ component momentum through the face perpendicular to the z direction with centre T is is v_θ times the mass flow through the face. The area of the face is

$$dr\,(r + ½dr)d\theta$$

so the rate of flow of mass through the face is

$$\rho v_z\, dr\,(r + ½dr)d\theta$$

and the rate of flow of θ component momentum through the face is

$$\rho v_\theta v_z\, dr\,(r + ½dr)d\theta$$

The corresponding rate of flow of θ component momentum out of the face with centre U is

$$\left(\rho v_\theta v_z + \frac{\partial(\rho v_\theta v_z)}{\partial z} dz\right) dr\,(r + ½dr)d\theta$$

so the net rate of flow of r component momentum out of the CV through the faces with centres T and U is

$$\frac{\partial(\rho v_\theta v_z)}{\partial z} dr\,(r + ½dr)d\theta\, dz = \frac{\partial(\rho v_\theta v_z)}{\partial z} dr\, rd\theta\, dz + \frac{\partial(\rho v_\theta v_z)}{\partial z} ½dr^2\, d\theta\, dz$$

We can neglect the term in dr^2, so the net rate of flow of θ component momentum out of the CV is

$$\frac{\partial(\rho v_\theta v_z)}{\partial z} dr\, rd\theta\, dz \qquad (2.4)$$

Adding together (2.2), (2.3) and (2.4), the sum of the net rates of outflow of x component momentum is

$$\left[\frac{\partial(\rho v_r v_\theta)}{\partial r} + \frac{\rho v_r v_\theta}{r} + \frac{1}{r}\frac{\partial(\rho {v_\theta}^2)}{\partial \theta} + \frac{\partial(\rho v_\theta v_z)}{\partial z}\right] dr\ rd\theta\ dz$$

Finally, we can combine the first and second terms in the brackets into one:

$$\left[\frac{1}{r}\frac{\partial(r\rho v_r v_\theta)}{\partial r} + \frac{1}{r}\frac{\partial(\rho {v_\theta}^2)}{\partial \theta} + \frac{\partial(\rho v_\theta v_z)}{\partial z}\right] dr\ rd\theta\ dz \qquad (2.5)$$

2.2 Body force terms

A body force is a vector, so in general it has three components f_r, f_θ, f_z per unit mass. The body force acting in the θ coordinate direction is

$$\rho f_\theta\ dr\ rd\theta\ dz \qquad (2.6)$$

2.3 Pressure term

Referring to Figure 2, the net force in the θ direction due to the pressure is

$$\left(p - \frac{1}{2}\frac{\partial p}{\partial \theta} d\theta\right) dr\ dz - \left(p + \frac{1}{2}\frac{\partial p}{\partial \theta} d\theta\right) dr\ dz$$

$$= -\frac{1}{r}\frac{\partial p}{\partial \theta} dr\ rd\theta\ dz \qquad (2.7)$$

2.4 Viscous stress terms

2.4.1 $\sigma_{\theta\theta}$ stress

Referring to Figure 3, the net force in the θ direction due to the stress $\sigma_{\theta\theta}$ is

$$\left(\sigma_{\theta\theta}+\frac{\partial\sigma_{\theta\theta}}{\partial\theta}d\theta\right)dr\,dz\cos\frac{d\theta}{2}-\sigma_{\theta\theta}dr\,dz\cos\frac{d\theta}{2}$$

For small θ we can approximate $\cos\theta$ as $1-\theta$ in radians. The net force in the θ direction is then

$$\sigma_{\theta\theta}\,dr\,dz-\sigma_{\theta\theta}\,dr\,dz\frac{d\theta}{2}+\frac{\partial\sigma_{\theta\theta}}{\partial\theta}d\theta\,dr\,dz-\frac{\partial\sigma_{\theta\theta}}{\partial\theta}d\theta\,dr\,dz\frac{d\theta}{2}-\sigma_{\theta\theta}dr\,dz+\sigma_{\theta\theta}dr\,dz\frac{d\theta}{2}$$

$$=\frac{\partial\sigma_{\theta\theta}}{\partial\theta}dr\,d\theta\,dz-\frac{\partial\sigma_{\theta\theta}}{\partial\theta}dr\,dz\frac{d\theta^2}{2}$$

We can neglect the term in $d\theta^2$, so the net force in the θ direction due to the stress $\sigma_{\theta\theta}$ is

$$\frac{\partial\sigma_{\theta\theta}}{\partial\theta}dr\,d\theta\,dz \qquad (2.8)$$

2.4.2 $\sigma_{\theta r}$ stress

The net force in the θ direction due to the stress $\sigma_{\theta r}$ is

$$\left(\sigma_{\theta r}+\frac{\partial\sigma_{\theta r}}{\partial r}dr\right)(r+dr)d\theta\,dz-\sigma_{\theta r}rd\theta\,dz$$

$$=\sigma_{\theta r}\,dr\,d\theta\,dz+\frac{\partial\sigma_{\theta r}}{\partial r}dr\,rd\theta\,dz\;+\frac{\partial\sigma_{\theta r}}{\partial r}dr^2\,d\theta\,dz$$

We can neglect the term in dr^2, so the net force in the r direction due to the stress $\sigma_{\theta r}$ is

$$\sigma_{\theta r}\,dr\,d\theta\,dz+\frac{\partial\sigma_{\theta r}}{\partial r}dr\,rd\theta\,dz \qquad (2.9)$$

2.4.3 $\sigma_{\theta z}$ stress

The net force in the θ direction due to the stress $\sigma_{\theta z}$ is

$$\left(\sigma_{\theta z}+\frac{\partial\sigma_{\theta z}}{\partial z}dz\right)dr\,(r+½dr)d\theta-\sigma_{\theta z}dr\,(r+½dr)d\theta$$

$$=\frac{\partial\sigma_{\theta z}}{\partial z}dr\,rd\theta\,dz+\frac{\partial\sigma_{\theta z}}{\partial z}½dr^2\,d\theta\,dz$$

We can neglect the term in dr^2, so the net force in the θ direction due to the stress $\sigma_{\theta z}$ is

$$\frac{\partial\sigma_{\theta z}}{\partial z}dr\,rd\theta\,dz \qquad (2.10)$$

2.4.4 $\sigma_{r\theta}$ stress

Referring to Figure 3, the net force in the θ direction due to the stress $\sigma_{r\theta}$ is

$$\left(\sigma_{r\theta} + \frac{\partial \sigma_{r\theta}}{\partial \theta} d\theta\right) dr\ dz \sin\frac{d\theta}{2} + \sigma_{r\theta}\ dr\ dz \sin\frac{d\theta}{2}$$

For small θ we can approximate sinθ as θ in radians. With this approximation the net force is

$$\sigma_{r\theta} dr\ dz \frac{d\theta}{2} + \frac{\partial \sigma_{r\theta}}{\partial \theta} dr\ dz \frac{d\theta^2}{2} + \sigma_{r\theta}\ dr\ dz \frac{d\theta}{2}$$

$$= \sigma_{r\theta}\ dr\ d\theta\ dz + \frac{\partial \sigma_{r\theta}}{\partial \theta} dr\ dz \frac{d\theta^2}{2}$$

We can neglect the term in $d\theta^2$, so the net force in the θ direction due to the stress $\sigma_{r\theta}$ is

$$\sigma_{r\theta}\ dr\ d\theta\ dz \qquad (2.11)$$

Adding together the expressions (2.8), (2.9), (2.10) and (2.11), the sum of the net forces in the θ direction due to the viscous stresses is

$$\left[\frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\sigma_{\theta r}}{r} + \frac{\partial \sigma_{\theta r}}{\partial r} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{\sigma_{r\theta}}{r}\right] dr\ r d\theta\ dz \qquad (2.12)$$

2.5 θ component momentum equation in terms of stress

By substituting expressions (2.1), (2.5), (2.6), (2.7) and (2.12) into Eq. (1.1) and dividing by the volume of the CV, $dr\ rd\theta\ dz$, we obtain the r component momentum equation in terms of stress:

$$\frac{\partial(\rho v_\theta)}{\partial t}+\frac{1}{r}\frac{\partial(r\rho v_r v_\theta)}{\partial r}+\frac{1}{r}\frac{\partial(\rho {v_\theta}^2)}{\partial \theta}+\frac{\partial(\rho v_\theta v_z)}{\partial z}$$

$$=-\frac{1}{r}\frac{\partial p}{\partial \theta}+\frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta}+\frac{\sigma_{\theta r}}{r}+\frac{\partial \sigma_{\theta r}}{\partial r}+\frac{\partial \sigma_{\theta z}}{\partial z}+\frac{\sigma_{r\theta}}{r}+\rho f_\theta \qquad (2.13)$$

2.6 θ component momentum equation in terms of velocity

Substituting the equations for $\sigma_{\theta\theta}$, $\sigma_{\theta r}$, $\sigma_{\theta z}$, $\sigma_{r\theta}$ into the Eq. (2.13) gives the θ component momentum equation for three-dimensional unsteady compressible flow in cylindrical coordinates:

$$\frac{\partial(\rho v_\theta)}{\partial t}+\frac{1}{r}\frac{\partial(r\rho v_r v_\theta)}{\partial r}+\frac{1}{r}\frac{\partial(\rho {v_\theta}^2)}{\partial \theta}+\frac{\partial(\rho v_\theta v_z)}{\partial z}$$

$$=-\frac{1}{r}\frac{\partial p}{\partial \theta}+\frac{1}{r}\frac{\partial}{\partial \theta}\left[2\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r}{r}\right)+\lambda\nabla\cdot\mathbf{V}\right]+\frac{\partial}{\partial r}\left[\mu\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta}+\frac{\partial v_\theta}{\partial r}-\frac{v_\theta}{r}\right)\right]$$

$$+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v_\theta}{\partial z}+\frac{1}{r}\frac{\partial v_z}{\partial \theta}\right)\right]+\frac{2\mu}{r}\left[\frac{1}{r}\frac{\partial v_r}{\partial \theta}+\frac{\partial v_\theta}{\partial r}-\frac{v_\theta}{r}\right]+\rho f_\theta \qquad (2.14)$$

3 z component momentum equation

3.1 Transient and convection terms

The z component of momentum in the CV of Figure 1 is equal to the z component of velocity, v_z, times the mass of fluid in the CV, $\rho\ dr\ rd\theta\ dz$; that is, $\rho\ v_z\ dr\ rd\theta\ dz$. The rate of increase of z component momentum with time (the first term in Eq. (1.1)), is therefore

$$\frac{\partial(\rho v_z)}{\partial t} dr\ rd\theta\ dz \qquad (3.1)$$

The rate of flow of z component momentum through the face perpendicular to the r direction whose centre is P is v_z times the mass flow through the face, $\rho\ v_r\ rd\theta\ dz$; that is,

$$\rho v_r v_z\ rd\theta\ dz$$

The rate of flow of z component momentum through the opposite face whose centre is Q is

$$\left(\rho v_r v_z + \frac{\partial(\rho v_r v_z)}{\partial r} dr\right)(r + dr)d\theta\ dz$$

and so the net rate of flow of z component momentum out of the CV through the faces with centres P and Q is

$$\left(\rho v_r v_z + \frac{\partial(\rho v_r v_z)}{\partial r} dr\right)(r + dr)d\theta\ dz - (\rho v_r v_z)\ rd\theta\ dz =$$

$$\frac{\partial(\rho v_r v_z)}{\partial r} dr\ rd\theta\ dz + \rho v_r v_z dr\ d\theta\ dz + \frac{\partial(\rho v_r v_z)}{\partial r} dr^2\ d\theta\ dz$$

We can neglect the term in dr^2, so the net rate of flow of z component momentum out of the CV through the faces with centres P and Q is

$$\frac{\partial(\rho v_r v_z)}{\partial r} dr\ rd\theta\ dz + \rho v_r v_z\ dr\ d\theta\ dz \qquad (3.2)$$

The rate of flow of z component momentum through the face perpendicular to the θ direction whose centre is R is v_z times the mass flow through the face, $\rho\ v_\theta\ dr\ dz$; that is,

$$\rho v_\theta v_z dr\ dz$$

The corresponding rate of flow of z component momentum out of the face with centre S is

$$\left(\rho v_\theta v_z + \frac{\partial(\rho v_\theta v_z)}{\partial\theta} d\theta\right) dr\ dz$$

so the net rate of flow of z component momentum out of the CV through the faces with centres R and S is

$$\frac{\partial(\rho v_\theta v_z)}{\partial\theta} dr\ d\theta\ dz \qquad (3.3)$$

The rate of flow of z component momentum through the face perpendicular to the z direction with centre T is is v_z times the mass flow through the face. The area of the face is

$$dr\ (r + ½dr)d\theta$$

so the rate of flow of mass through the face is

$$\rho v_z\ dr\ (r + ½dr)d\theta$$

and the rate of flow of z component momentum through the face is

$$\rho {v_z}^2\ dr\ (r + ½dr)d\theta$$

The corresponding rate of flow of θ component momentum out of the face with centre U is

$$\left(\rho {v_z}^2 + \frac{\partial(\rho {v_z}^2)}{\partial z} dz\right) dr\ (r + ½dr)d\theta$$

so the net rate of flow of r component momentum out of the CV through the faces with centres T and U is

$$\frac{\partial(\rho {v_z}^2)}{\partial z} dr\ (r + ½dr)d\theta\ dz$$

$$= \frac{\partial(\rho {v_z}^2)}{\partial z} dr\ rd\theta\ dz + \frac{\partial(\rho {v_z}^2)}{\partial z} ½dr^2\ d\theta\ dz$$

We can neglect the term in dr^2, so the net rate of flow of z component momentum out of the CV is

$$\frac{\partial(\rho {v_z}^2)}{\partial z} dr\ rd\theta\ dz \qquad (3.4)$$

Adding together (3.2), (3.3) and (3.4), the sum of the net rates of outflow of x component momentum is

$$\left[\frac{\partial(\rho v_r v_z)}{\partial r} + \frac{\rho v_r v_z}{r} + \frac{1}{r}\frac{\partial(\rho v_\theta v_z)}{\partial \theta} + \frac{\partial(\rho {v_z}^2)}{\partial z}\right] dr\ rd\theta\ dz$$

Finally, we can combine the first and second terms in the brackets into one:

$$\left[\frac{1}{r}\frac{\partial(r\rho v_r v_z)}{\partial r} + \frac{1}{r}\frac{\partial(\rho v_\theta v_z)}{\partial \theta} + \frac{\partial(\rho {v_z}^2)}{\partial z}\right] dr\ rd\theta\ dz \qquad (3.5)$$

3.2 Body force terms

A body force is a vector, so in general it has three components f_r, f_θ, f_z per unit mass. The body force acting in the z coordinate direction is

$$\rho f_z\ dr\ rd\theta\ dz \qquad (3.6)$$

3.3 Pressure term

Referring to Figure 2, the net force in the z direction due to the pressure is

$$\left(p - \frac{1}{2}\frac{\partial p}{\partial z}dz\right) dr\ (r + ½dr)d\theta - \left(p + \frac{1}{2}\frac{\partial p}{\partial z}dz\right) dr\ (r + ½dr)d\theta$$

$$= -\frac{\partial p}{\partial z} r\ rd\theta\ dz - \frac{\partial p}{\partial z} ½dr^2\ d\theta\ dz$$

After neglecting the dr^2 term, the net force is

$$-\frac{\partial p}{\partial z} dr\ rd\theta\ dz \qquad (3.7)$$

3.4 Viscous stress terms

3.4.1 σ_{zr} stress

Referring to Figure 3, the net force in the z direction due to the stress σ_{zr} is

$$\left(\sigma_{zr} + \frac{\partial \sigma_{zr}}{\partial r} dr\right)(r + dr) d\theta\, dz - \sigma_{zr} r d\theta\, dz$$

$$= \sigma_{zr}\, dr\, d\theta\, dz + \frac{\partial \sigma_{zr}}{\partial r} dr\, r d\theta\, dz\, + \frac{\partial \sigma_{zr}}{\partial r} dr^2\, d\theta\, dz$$

After neglecting the term in dr^2, we have

$$\sigma_{zr}\, dr\, d\theta\, dz + \frac{\partial \sigma_{zr}}{\partial r} dr\, r d\theta\, dz$$

We can combine these two terms into one. The net force in the z direction due to the stress σ_{zr} is therefore

$$\frac{1}{r}\frac{\partial r \sigma_{zr}}{\partial r} dr\, r d\theta\, dz \qquad (3.8)$$

3.4.2 $\sigma_{z\theta}$ stress

The net force in the z direction due to the stress $\sigma_{z\theta}$ is

$$\left(\sigma_{z\theta} + \frac{\partial \sigma_{z\theta}}{\partial \theta} d\theta\right) dr\, dz - \sigma_{z\theta} dr\, dz$$

$$= \frac{\partial \sigma_{z\theta}}{\partial \theta} dr\, d\theta\, dz \qquad (3.9)$$

3.4.3 σ_{zz} stress

The net force in the z direction due to the stress σ_{zz} is

$$\left(\sigma_{zz} + \frac{\partial \sigma_{zz}}{\partial z} dz\right) dr\, (r + ½dr) d\theta - \sigma_{zz} dr\, (r + ½dr) d\theta$$

$$= \frac{\partial \sigma_{zz}}{\partial z} dr\, r d\theta\, dz + \frac{\partial \sigma_{zz}}{\partial z} ½dr^2 d\theta\, dz$$

We can neglect the term in dr^2, so the net force in the z direction due to the stress σ_{zz} is

$$\frac{\partial \sigma_{zz}}{\partial z} dr\, r d\theta\, dz \qquad (3.10)$$

Adding together the expressions (3.8), (3.9) and (3.10), the sum of the net forces in the z direction due to the viscous stresses is

$$\left[\frac{1}{r}\frac{\partial r \sigma_{zr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{z\theta}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z}\right] dr\, r d\theta\, dz \qquad (3.11)$$

3.5 z component momentum equation in terms of stress

By substituting expressions (3.1), (3.5), (3.6), (3.7) and (3.11) into Eq. (1.1) and dividing by the volume of the CV, $dr\ rd\theta\ dz$, we obtain the r component momentum equation in terms of stress:

$$\frac{\partial(\rho v_z)}{\partial t}+\frac{1}{r}\frac{\partial(r\rho v_r v_z)}{\partial r}+\frac{1}{r}\frac{\partial(\rho v_\theta v_z)}{\partial \theta}+\frac{\partial(\rho {v_z}^2)}{\partial z}$$

$$=-\frac{\partial p}{\partial z}+\frac{1}{r}\frac{\partial r\sigma_{zr}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{z\theta}}{\partial \theta}+\frac{\partial \sigma_{zz}}{\partial z}+\rho f_z \qquad (3.12)$$

3.6 z component momentum equation in terms of velocity

Substituting the equations for σ_{zr}, $\sigma_{z\theta}$, σ_{zz} into the Eq. (3.12) gives the z component momentum equation for three-dimensional unsteady compressible flow in cylindrical coordinates:

$$\frac{\partial(\rho v_z)}{\partial t}+\frac{1}{r}\frac{\partial(r\rho v_r v_z)}{\partial r}+\frac{1}{r}\frac{\partial(\rho v_\theta v_z)}{\partial \theta}+\frac{\partial(\rho {v_z}^2)}{\partial z}$$

$$=-\frac{\partial p}{\partial z}+\frac{1}{r}\frac{\partial}{\partial r}\left[\mu r\left(\frac{\partial v_r}{\partial z}+\frac{\partial v_z}{\partial r}\right)\right]+\frac{1}{r}\frac{\partial}{\partial \theta}\left[\mu\left(\frac{\partial v_\theta}{\partial z}+\frac{1}{r}\frac{\partial v_z}{\partial \theta}\right)\right]$$

$$+\frac{\partial}{\partial z}\left[2\mu\frac{\partial v_z}{\partial z}+\lambda\nabla\cdot\mathbf{V}\right]+\rho f_z \qquad (3.13)$$

4 References

1. M. R. Spiegel, *Vector Analysis and an Introduction to Tensor Analysis*, Schaum's Outline Series, McGraw-Hill, 1959.